ON THE $\mathfrak{S}_n$-MODULE STRUCTURE OF THE NONCOMMUTATIVE HARMONICS.

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Abstract. Using a noncommutative analog of Chevalley's decomposition of polynomials into symmetric polynomials times coinvariants due to Bergeron, Reutenauer, Rosas, and Zabrocki we compute the graded Frobenius characteristic for their two sets of noncommutative harmonics with respect to the left action of the symmetric group (acting on variables). We use these results to derive the Frobenius series for the enveloping algebra of the derived free Lie algebra in n variables.

In honor of Manfred Schocker (1970-2006). The authors would also like to acknowledge the contributions that he made to this paper.

1. Introduction

A central result of Claude Chevalley [3] decomposes the ring of polynomials in n variables (as graded representation of the symmetric group $\mathfrak{S}_n$) as the tensor product of the symmetric polynomials times the coinvariants of $\mathfrak{S}_n$ (i.e., polynomials modulo symmetric polynomials with no constant term).

The coinvariants of the symmetric group can also be defined as its harmonics (the polynomials annhilated by all symmetric polynomial differential operators with no constant term). They admit as a basis the famous Schubert polynomials of Schubert calculus, that play an important role in algebraic combinatorics, see for instance [6].

The space of invariant polynomials in noncommutative variables was introduced in 1936 by Wolf [16] where she found a noncommutative version of the fundamental theorem of symmetric functions. This space has been studied from a modern perspective in [13, 1, 2]. On the other hand, two sets of noncommutative harmonics for the symmetric group were introduced in [1] that translated into two noncommutative analogues of Chevalley decomposition for the ring of polynomials in noncommuting variables. The question of decomposing as $\mathfrak{S}_n$–modules both kinds of noncommutative harmonics was left open. This is the starting point in our investigations.

We begin the present work with the computation of the graded Frobenius characteristic of noncommutative harmonics. We then use these calculations

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to derive the Frobenius series for the enveloping algebra of the derived free Lie algebra in n variables, $\mathcal{A}'_n$. This last computation is achieved by using the existence of an isomorphism of $GL_n(\mathbb{Q})$–modules between the space of polynomials in noncommutative variables, and the tensor product of the space of commuting polynomials with $\mathcal{A}'_n$. Such an isomorphism is presented explicitly in the last section.

We conclude this introduction with some basic definitions and results that we will be using in the following sections. Let $\mathfrak{S}_n$ denote the symmetric group in n letters. Denote by $\mathbb{Q}[X_n] = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ the space of polynomials in n commuting variables and by $\mathbb{Q}\langle X_n\rangle = \mathbb{Q}\langle x_1, x_2, \ldots, x_n\rangle$ the space of polynomials in n noncommutative variables.

The space of symmetric polynomials in n variables will be denoted by Sym_n and the space of noncommutative polynomials which are invariant under the canonical action of the symmetric group $\mathfrak{S}_n$ will be denoted by $NCSym_n$.

Given any polynomial $f(X_n) \in \mathbb{Q}[X_n]$, the notation $f(\partial_{X_n})$ represents the polynomial turned into an operator with each of the variables replaced by its corresponding derivative operator. Analogous notation will also hold for $f(X_n) \in \mathbb{Q}\langle X_n\rangle$ except that there are two types of differential operators acting on words in noncommutative variables. The first is the *Hausdorff derivative*, ∂_x, whose action on a word w is defined to be the sum of the subwords of w with an occurrence of the letter x deleted. The second derivative is the *twisted derivative*, d_x, which is defined on w to be w' if $w = xw'$, and 0 otherwise. Both derivations are extended to polynomials by linearity.

It is interesting to remark (as does Lenormand in [8], section *Séries comme opérateurs*) that these two operations are dual to the shuffle and concatenation products respectively, with respect to a scalar product where the noncommutative monomials are self dual. That is,

$$\begin{aligned} \langle \partial_x f, g\rangle &= \langle f, x \sqcup\!\sqcup g\rangle, \text{ and} \\ \langle d_x f, g\rangle &= \langle f, xg\rangle. \end{aligned}$$

Following [1], we introduce the following two sets of noncommutative analogues of the harmonic polynomials. The canonical action of the symmetric group endow them with the structure of $\mathfrak{S}_n$–modules.

$$\begin{aligned} MHar_n &= \{f \in \mathbb{Q}\langle X_n\rangle : \ p(\partial_{X_n})f(X_n) = 0 \text{ for all } p \in \mathfrak{M}_n\} \\ NCHar_n &= \{f \in \mathbb{Q}\langle X_n\rangle : \ p(d_{X_n})f(X_n) = 0 \text{ for all } p \in \mathfrak{M}_n\} \end{aligned}$$

where $\mathfrak{M}_n = \{p \in NCSym_n \text{ with } p(0) = 0\}$.

We are now ready to state the two decompositions of $\mathbb{Q}\langle X_n\rangle$ as the tensor product (over $\mathbb{Q}$) of its invariants times its coinvariants that we have described.

Proposition 1 ([1], Theorems 6.8 and 8.8)**.** *As graded $\mathfrak{S}_n$–modules,*

$$\begin{aligned} \mathbb{Q}\langle X_n\rangle &\simeq MHar_n \otimes Sym_n, \\ \mathbb{Q}\langle X_n\rangle &\simeq NCHar_n \otimes NCSym_n. \end{aligned}$$

2. The Frobenius characteristic of noncommutative harmonics

In this section we compute the Frobenius characteristic of both kinds of noncommutative harmonics. This section is based of the observation that the graded Frobenius series for each of the $\mathfrak{S}_n$–modules appearing in Proposition 1 is either known or can be deduced from the existence of the isomorphisms described there.

The expressions for Frobenius images and characters will require a little use of symmetric function notation and identities. We will follow Macdonald [9] for the notation of the s_λ Schur, h_λ homogeneous, e_λ elementary and p_λ power sums bases for the ring of symmetric functions Sym, that we identify with $\mathbb{Q}[p_1, p_2, p_3, \ldots]$. For convenience we will make use of some plethystic notation.

For a symmetric function f, $f[X]$ represents the symmetric function evaluated at an unspecified (possibly infinite) alphabet X. Then, $f[X(1-q)]$ is the image of f under the algebra automorphism sending the power sum symmetric function p_k to $(1-q^k)p_k[X]$. Similarly, $f\left[\frac{X}{1-q}\right]$ is the image of the symmetric function f under the inverse automorphism (sending the power sum p_k to $p_k/(1-q^k)$).

In our calculations, we use the Kronecker product $\odot$ of symmetric functions. This operation on symmetric functions corresponds, under the Frobenius map, to the inner tensor product of representations of the symmetric group (tensor product of representations with the diagonal action on the tensors). It can also be defined directly on symmetric functions by the equation $p_\lambda \odot p_\mu = \delta_{\lambda,\mu}\left(\prod_i n_i(\lambda)! i^{n_i(\lambda)}\right) p_\lambda$ where $n_i(\lambda)$ is the number of parts of size i in λ, and then extended by bilinearity.

We introduce the notations

$$\begin{aligned} (q;q)_k &= (1-q)(1-q^2)\cdots(1-q^k), \\ \{q;q\}_k &= (1-q)(1-2q)\cdots(1-k\,q). \end{aligned}$$

Then $q^d/\{q;q\}_d$ is the generating function for the set partitions with length d and $q^d/(q;q)_d$ is the generating function for partitions with length d, [15]. Finally, since Sym_n and $NCSym_n$ are made of graded copies of the trivial $\mathfrak{S}_n$-module we conclude that

$$\begin{aligned} \mathcal{F}rob_{\mathfrak{S}_n}(NCSym_n) &= h_n[X]\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d} \\ \mathcal{F}rob_{\mathfrak{S}_n}(Sym_n) = h_n[X]\frac{1}{(q;q)_n} &= h_n[X]\sum_{d=0}^{n}\frac{q^d}{(q;q)_d}. \end{aligned}$$

In the following lemma we compute the graded Frobenius characteristic for the module $\mathbb{Q}\langle X_n\rangle$.

Lemma 2 (The Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$)**.**

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) = \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X].$$

Proof. For each monomial $x_{i_1}\cdots x_{i_r}$, we define its *type* $\nabla(x_{i_1}\cdots x_{i_r})$ to be the set partition of $[r]=\{1,2,\ldots,r\}$ such that a and b are in the same part of the set partition if and only if $i_a = i_b$ in the monomial. For a set partition A with at most n parts, we will let M^A equal the $\mathfrak{S}_n$ submodule of $\mathbb{Q}\langle X_n\rangle$ spanned by all monomials of type A. As $\mathfrak{S}_n$–module,

$$\mathbb{Q}\langle X_n\rangle \simeq \bigoplus_{d=0}^{n} \bigoplus_{A\,:\,\ell(A)=d} M^A$$

where the second direct sum is taken over all set partitions A with d parts.

Fix a set partition A, and let d be the number of parts of A, and $\mathbf{x}_{\vec{i}} = x_{i_1}x_{i_2}...x_{i_r}$ be the smallest monomial in lex order in M^A. It involves only the variables x_1, x_2, …, x_d. The representation M^A is the representation of $\mathfrak{S}_n$ induced by the action of the subgroup $\mathfrak{S}_d \times \mathfrak{S}_1^{n-d} \simeq \mathfrak{S}_d$ on the subspace $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_{\vec{i}}$. The representation $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_{\vec{i}}$ of $\mathfrak{S}_d$ is isomorphic to the regular representation. We use the rule for a representation R of $\mathfrak{S}_d$ induced to $\mathfrak{S}_n$,

$$\mathcal{F}rob_{\mathfrak{S}_n}(R\uparrow_{\mathfrak{S}_d}^{\mathfrak{S}_n}) = h_{n-d}[X]\mathcal{F}rob_{\mathfrak{S}_d}(R),$$

and conclude that the Frobenius characteristic of M^A is $h_{(n-d,1^d)}[X]$. Hence the graded Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$ is

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) = \sum_{d=0}^{n}\sum_{A:\ell(A)=d} q^{|A|} h_{(n-d,1^d)}[X] = \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X].$$

□

We are now able to compute the Frobenius characteristic for $MHar_n$ and $NCHar_n$.

Theorem 3 (The Frobenius characteristic of the noncommutative harmonics)**.**

$$\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n) = (q;q)_n \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X]$$

and

$$\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) = \Big(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d}\Big)^{-1} \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X].$$

Proof. This follows since $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n \otimes Sym_n) = \mathcal{F}rob_{\mathfrak{S}_n}(MHar_n) \odot \mathcal{F}rob_{\mathfrak{S}_n}(Sym_n)$. Since $h_n[X]$ is the unity for the Kronecker product on symmetric functions of degree n, and since $\mathcal{F}rob_{\mathfrak{S}_n}(Sym_n) = h_n[X]/(q;q)_n$, we conclude that $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n)/(q;q)_n = \mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle)$. We can now solve for $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n)$.

A similar argument demonstrates the formula for $\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)$. We have from Proposition 1 and Lemma 2,

$$\begin{aligned}
\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X] &= \mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) \\
&= \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) \odot \mathcal{F}rob_{\mathfrak{S}_n}(NCSym_n) \\
&= \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_n[X] \odot \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) \\
&= \left(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \right) \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n).
\end{aligned}$$

From this equation we can solve for $\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)$. □

As a corollary, we obtain the generating functions for the graded dimensions of these spaces.

Corollary 4 (The Hilbert series of the noncommutative harmonics)**.**

$$\begin{aligned}
dim_q(MHar_n) &= \frac{(q;q)_n}{1-nq} \\
dim_q(NCHar_n) &= \frac{1}{(1-nq)\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d}}
\end{aligned}$$

Proof. After Theorem 3,

$$\begin{aligned}
\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n) &= (q;q)_n \, \mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) \\
\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) &= \Big(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \Big)^{-1} \mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle)
\end{aligned}$$

This implies

$$\begin{aligned}
dim_q(MHar_n) &= (q;q)_n \, dim_q(\mathbb{Q}\langle X_n\rangle) \\
dim_q(NCHar_n) &= \Big(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \Big)^{-1} dim_q(\mathbb{Q}\langle X_n\rangle)
\end{aligned}$$

since the Hilbert series of a graded $\mathfrak{S}_n$–module is obtained by coefficient extraction from the graded Frobenius characteristic (of the coefficient of $p_{(1^n)}[X]/n!$ in the expansion in power sum symmetric functions). Last, the Hilbert series of $\mathbb{Q}\langle X_n\rangle$ is $\frac{1}{1-nq}$. □

The graded dimensions of $MHar_n$ for $2 \le n \le 5$ are listed in [14] as sequences $A122391$ through $A122394$. The sequences of graded dimensions of $NCHar_n$ for $3 \le n \le 8$ are listed in [14] as sequences $A122367$ through $A122372$.

3. Non–commutative harmonics and the enveloping algebra of the derived free Lie algebra

Let $\mathcal{L}_n$ be the canonical realization of the free Lie algebra inside the ring of polynomials in noncommuting variables $\mathbb{Q}\langle X_n\rangle$. More precisely, $\mathcal{L}_n$ is the linear span of the minimal set of polynomials in $\mathbb{Q}\langle X_n\rangle$ that includes $\mathbb{Q}$ and the variables X_n, and is closed under the bracket operation $[x, y] = xy - yx$. Let $\mathcal{L}'_n = [\mathcal{L}_n, \mathcal{L}_n]$ be the derived free Lie algebra. Remark that $\mathcal{L}_n = \mathcal{L}'_n \oplus \mathbb{Q}X_n$, where $\mathbb{Q}X_n$ denotes the space of linear polynomials. The enveloping algebra $\mathcal{A}'_n$ of $\mathcal{L}_n$ can be realized as a subalgebra of $\mathbb{Q}\langle X_n\rangle$ as follows (see [12] 1.6.5):

$$\mathcal{A}'_n = \bigcap_{x\in X_n} \ker \partial_x.$$

More explicitly, $\mathcal{A}'_n$ is the subalgebra of $\mathbb{Q}\langle X_n\rangle$ generated by all the brackets under concatenation.

In [1] it was established that there is an isomorphism of vector spaces between $MHar_n$ and $\mathcal{A}'_n \otimes \mathcal{H}_n$. In this section we will show the following result.

Theorem 5. *As $\mathfrak{S}_n$–modules,*

$$MHar_n \simeq \mathcal{A}'_n \otimes \mathcal{H}_n.$$

The proposition will be established by comparing the Frobenius image of $MHar_n$ (known from Theorem 3) to $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n \otimes \mathcal{H}_n)$, which is equal to $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n) \odot \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{H}_n)$. We will determine $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n)$ in Theorem 8 below. An intermediate step will make use the following Theorem due to V. Drensky.

Proposition 6 (Drensky, [5] Theorem 2.6)**.** *As $GL_n(\mathbb{Q})$–modules (and consequently as $\mathfrak{S}_n$–modules),*

$$\mathbb{Q}\langle X_n\rangle \simeq \mathbb{Q}[X_n] \otimes \mathcal{A}'_n.$$

Drensky proved Proposition 6 by exhibiting an explicit isomorphism between these two representations. We will look at it in the next section. For now, we will provide a non–constructive proof of the theorem. Before, we need to introduce some notation.

It is known that $\mathbb{Q}\langle X_n\rangle$ is the universal enveloping algebra (u.e.a) of the free Lie algebra, $\mathcal{L}_n$. Using the Poincaré-Birkhoff-Witt theorem, a linear basis for $\mathbb{Q}\langle X_n\rangle$ is given by decreasing products of elements of $\mathcal{L}_n$. Since we can choose an ordering of the elements of $\mathcal{L}_n$ so that the space of linear polynomials is smallest and decreasing products of linear polynomials are isomorphic to $\mathbb{Q}[X_n]$ (as a vector space), we note that as vector spaces

$$\mathbb{Q}\langle X_n\rangle = u.e.a.(\mathcal{L}_n) = u.e.a(\mathbb{Q}X_n \oplus \mathcal{L}'_n) \simeq \mathbb{Q}[X_n] \otimes \mathcal{A}'_n.$$

To distinguish between the commutative elements of $\mathbb{Q}[X_n]$ and the non-commutative words of $\mathbb{Q}\langle X_n\rangle$, we will place a dot over the variables (as in $\dot{x}_i$) to indicate the commutative variables.

Let $[n] = \{1, 2, \ldots, n\}$ and let $[n]^r$ denote the words of length r in the alphabet of the numbers $1, 2, \ldots, n$. A word $w \in [n]^r$ is called a Lyndon word if $w < w_k w_{k+1} \cdots w_r$ for all $2 \leq k \leq r$ where $<$ represents lexicographic order on words.

Every word $w \in [n]^r$ is equal to a unique product $w = \ell_1 \ell_2 \cdots \ell_k$ such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ and each ℓ_i is Lyndon (e.g. Corollary 4.4 of [12]).

Let ℓ be a Lyndon word of length greater than 1. We say that $\ell = uv$ is the standard factorization of ℓ if v is the smallest nontrivial suffix in lexicographic order. It follows that u and v are Lyndon words and $u < v$.

For a Lyndon word ℓ, if ℓ is a single letter a then define $P_a = x_a \in \mathbb{Q}\langle X_n \rangle$. If $\ell = uv$ is the standard factorization of ℓ, then $P_\ell = [P_u, P_v]$. For any $w \in [n]^r$ with Lyndon decomposition $w = \ell_1 \ell_2 \cdots \ell_k$, define

$$P_w = P_{\ell_1} P_{\ell_2} \cdots P_{\ell_k}.$$

The set $\{P_w\}_{w \in [n]^r}$ forms a basis for the noncommutative polynomials of degree r ([12], Theorem 5.1). The elements P_w with Lyndon decomposition $w = \ell_1 \ell_2 \cdots \ell_k$ such that each Lyndon factor has degree at least 2 are a basis of $\mathcal{A}'_n$.

Proof. To prove that $\mathbb{Q}\langle X_n \rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are isomorphic as $GL_n(\mathbb{Q})$–modules, we use the fact that two polynomial $GL_n(\mathbb{Q})$–modules with the same character are isomorphic (see for instance the notes by Kraft and Procesi, [7]). The character of a $GL_n(\mathbb{Q})$–module is the trace of the action of the diagonal matrix $diag(a_1, a_2, \ldots, a_n)$.

A basis for $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are the elements $\dot{\mathbf{x}}^\alpha \otimes P_{\ell_1} \cdots P_{\ell_k}$ with $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ and $|\ell_i| \geq 2$. The action of the diagonal matrix $diag(a_1, a_2, \ldots, a_n)$ on this basis element is the same as the action on the noncommutative polynomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} P_{\ell_1} P_{\ell_2} \cdots P_{\ell_k}$ (in both cases: multiplication by $a_1^{\alpha_1+m_1} a_2^{\alpha_2+m_2} \cdots a_n^{\alpha_n+m_n}$ where m_i is the number of occurrences of i in the word $\ell_1 \ell_2 \cdots \ell_k$). By the Poincaré-Birkhoff-Witt theorem, these polynomials form a basis for $\mathbb{Q}\langle X_n \rangle$, hence the trace of the action of $diag(a_1, a_2, \ldots, a_n)$ acting on $\mathbb{Q}\langle X_n \rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are equal. Since their characters are equal, we conclude that they are isomorphic as $GL_n(\mathbb{Q})$ modules. □

The $GL_n(\mathbb{Q})$–character of $\mathbb{Q}[X_n]$ is $\prod_{i=1}^n \frac{1}{1-a_i}$, and the $GL_n(\mathbb{Q})$–character of $\mathbb{Q}\langle X_n \rangle$ is $\frac{1}{1-(a_1+a_2+\cdots+a_n)}$. Therefore, the existence of a $GL_n(\mathbb{Q})$-module isomorphism between $\mathbb{Q}\langle X_n \rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ implies the following result.

Corollary 7 (The $GL_n(\mathbb{Q})$–character of $\mathcal{A}'_n$)**.**

$$\begin{aligned} char_{GL_n(\mathbb{Q})}(\mathcal{A}'_n)(a_1, a_2, \cdots, a_n) &= \frac{(1-a_1)\cdots(1-a_n)}{1-(a_1+a_2+\cdots+a_n)} \\ &= \sum_{k \geq 0} \sum_{i=2}^{k} (-1)^i e_{(i,1^{k-i})}(a_1, a_2, \ldots, a_n). \end{aligned}$$

Moreover this last sum is equal to

$$\sum_T s_{shape(T)}(a_1, a_2, \ldots, a_n)$$

where the sum is over all standard tableaux T such that the smallest integer which does not appear in the first column of T is odd.

By Schur-Weyl duality, the above formula also describes the decomposition of the subspace of multilinear polynomials (i.e. with distinct occurrences of the variables) of $\mathcal{A}'_n$. That is, if n is the number of variables, the the multilinear polynomials of degree n will be an $\mathfrak{S}_n$-module with Frobenius image equal to $\sum_{i=2}^n (-1)^i e_{(i,1^{n-i})}[X]$. This decomposition was considered in the papers [4], [10], [11] where an expression was given degree by degree up to $n = 7$. The expansion of this formula in the Schur basis provided in the Theorem agrees with the computations in those papers.

We can derive a formula for the Frobenius characteristic of $\mathcal{A}'$ by using a similar technique.

Theorem 8 (The Frobenius characteristic of $\mathcal{A}'_n$)**.**

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n) = \sum_{d=0}^n \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X(1-q)].$$

Proof. For any symmetric function $f[X]$ of degree n, we have that

$$f[X] \odot h_n\left[\frac{X}{1-q}\right] = f\left[\frac{X}{1-q}\right].$$

In particular, since $\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}[X_n]) = h_n\left[\frac{X}{1-q}\right]$, we conclude that

$$\begin{aligned}
\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) &= \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n \otimes \mathbb{Q}[X_n]) \\
&= \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n) \odot h_n\left[\frac{X}{1-q}\right] = \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n)\left[\frac{X}{1-q}\right].
\end{aligned}$$

This implies that if we make the plethystic substitution $X \to X(1-q)$ into both sides of this equation and using Lemma 2 we arrive at the stated formula. □

We can now prove Theorem 5.

Proof. From Theorem 3 we know the Frobenius image of $MHar_n$, we compare this to

$$\begin{aligned}
\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n \otimes \mathcal{H}_n) &= \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n) \odot \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{H}_n), \\
&= \sum_{d=0}^n \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X(1-q)] \odot h_n\left[\frac{X}{1-q}\right](q;q)_n \\
&= (q;q)_n \sum_{d=0}^n \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X] \\
&= \mathcal{F}rob_{\mathfrak{S}_n}(MHar_n).
\end{aligned}$$

Since the two $\mathfrak{S}_n$–modules have the same Frobenius image, we conclude that they must be isomorphic. □

4. An explicit isomorphism between $\mathbb{Q}\langle X_n\rangle$ and $\mathbb{Q}[X_n]\otimes\mathcal{A}'_n$.

Let V be a finite–dimensional vector space over $\mathbb{Q}$. Let $S(V)$ and $T(V)$ be its symmetric algebra and tensor algebra respectively. There exists a unique embedding φ of $GL(V)$–modules of $S(V)$ into $T(V)$ such that

$$\varphi(v_1v_2\cdots v_r)=\sum_{\sigma\in\mathfrak{S}_r} v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(r)}\qquad \text{for all } r\geq 0,\ v_1,v_2,\ldots,v_r\in V.$$

Its image is the subspace of the symmetric tensors. In the case $V=\bigoplus_{i=1}^{n}\mathbb{Q}x_i$, we have $S(V)=\mathbb{Q}[X_n]$ and $T(V)=\mathbb{Q}\langle X_n\rangle$. Then the embedding φ and the inclusion $\mathcal{A}'_n\subset\mathbb{Q}\langle X_n\rangle$ induce a map of $GL_n(\mathbb{Q})$–modules $\Phi:\mathbb{Q}[X_n]\otimes\mathcal{A}'_n\longrightarrow\mathbb{Q}\langle X_n\rangle$ characterized by $\Phi(f\otimes a)=\varphi(f)a$ for all $f\in\mathbb{Q}[X_n]$ and all $a\in\mathcal{A}'_n$. Then,

Proposition 9 (Drensky, [5] Theorem 2.6)**.** *The map Φ is a $GL_n(\mathbb{Q})$ equivariant isomorphism from $\mathbb{Q}[X_n]\otimes\mathcal{A}'_n$ to $\mathbb{Q}\langle X_n\rangle$.*

Indeed, Drensky showed that given an arbitrary homogeneous basis of $\mathcal{G}$ of $\mathcal{A}'_n$, the elements $\Phi(m\otimes g)$ for m monomial and $g\in\mathcal{G}$, are a basis of $\mathcal{A}'_n$ ([5] Lemma 2.4). We refine Drensky's proof by considering for $\mathcal{G}$ the *bracket basis* $\{P_w\}_{w\in[n]^r}$ of $\mathcal{A}'_n$ (introduced before the proof of Proposition 6) and the *shuffle basis* (see below) to realize $\mathbb{Q}[X_n]$ in $\mathbb{Q}\langle X_n\rangle$. We show that the elements $\Phi(m\otimes g)$ form a basis $\mathbb{Q}\langle X_n\rangle$ (the *hybrid basis*) that is triangularly related and expands positively in the *bracket basis* of $\mathbb{Q}\langle X_n\rangle$ (Theorem 10 below).

We follow the book of Reutenauer [12] for the classical definitions and results used in this section. The bracket basis P_w has been introduced in the previous section (before the proof of Proposition 6). Before presenting the hybrid basis we introduce another classical basis of $\mathbb{Q}\langle X_n\rangle$: the *shuffle basis.*

The shuffle basis of $\mathbb{Q}\langle X_n\rangle$. Consider two monomials, $x_{i_1}x_{i_2}\cdots x_{i_r}$ and $x_{j_1}x_{j_2}\cdots x_{j_{r'}}$ in $\mathbb{Q}\langle X_n\rangle$. For a subset

$$S=\{s_1,s_2,\ldots,s_r\}\subseteq[r+r'],$$

and the complement subset $T=\{t_1,t_2,\ldots,t_{r'}\}=[r+r']\backslash S$, we let

$$x_{i_1}x_{i_2}\cdots x_{i_r}\sqcup\!\sqcup_S x_{j_1}x_{j_2}\cdots x_{j_{r'}}:=w$$

be the unique monomial in $\mathbb{Q}\langle X_n\rangle$ of length $r+r'$ such that $w_{s_1}w_{s_2}\cdots w_{s_r}=x_{i_1}x_{i_2}\cdots x_{i_r}$ and $w_{t_1}w_{t_2}\cdots w_{t_{r'}}=x_{j_1}x_{j_2}\cdots x_{j_{r'}}$.

The shuffle of any two monomials is defined as

$$u ⧢ v = \sum_{\substack{S\subseteq[|u|+|v|]\\|S|=|u|}} u ⧢_S v.$$

This shuffle of monomials is then extended to a bilinear operation on any two elements of $\mathbb{Q}\langle X_n\rangle$ The shuffle product is a commutative and associative operation on $\mathbb{Q}\langle X_n\rangle$.

Let w be a word in $[n]^r$ and let $w = \ell_1^{i_1}\ell_2^{i_2}\cdots\ell_k^{i_k}$ be the factorization of w into decreasing products of Lyndon words $\ell_1 > \ell_2 > \cdots > \ell_k$. For a Lyndon word $\ell = i_1 i_2 \cdots i_r$, let S_ℓ be the corresponding monomial in $\mathbb{Q}\langle X_n\rangle$, that is $S_\ell = x_{i_1}x_{i_2}\cdots x_{i_r}$. If w is not a single Lyndon word then define

$$S_w = \frac{1}{i_1!i_2!\cdots i_k!} S_{\ell_1}^{⧢ i_1} ⧢ S_{\ell_2}^{⧢ i_2} ⧢ \cdots ⧢ S_{\ell_k}^{⧢ i_k}.$$

The set $\{S_w\}_{w\in[n]^r}$ forms a basis for the noncommutative polynomials of degree r ([12], Corollary 5.5).

It is interesting to note that the bracket basis P_w and the shuffle basis S_w are dual with respect to the scalar product where the noncommutative monomials are self-dual.

The hybrid basis of $\mathbb{Q}\langle X_n\rangle$. We are now ready to introduce the *hybrid basis.*

Given a word $w \in [n]^r$ with a factorization into decreasing products of Lyndon words $w = \ell_1^{i_1}\ell_2^{i_2}\cdots\ell_k^{i_k}$, then let $\ell_{j_1}, \ell_{j_2}, \cdots, \ell_{j_r}$ be the Lyndon words of length 1 in this decomposition and set

$$M(w) = x_{\ell_{j_1}}^{⧢ i_{j_1}} ⧢ x_{\ell_{j_2}}^{⧢ i_{j_2}} ⧢ \cdots ⧢ x_{\ell_{j_r}}^{⧢ i_{j_r}} = i_{j_1}!i_{j_2}!\cdots i_{j_r}!S_{\ell_{j_1}^{i_{j_1}}\ell_{j_2}^{i_{j_2}}\cdots\ell_{j_r}^{i_{j_r}}}.$$

Observe that $M(w)$ is the image under the embedding φ of the monomial $X(w) = \dot{x}_{\ell_{j_1}}^{i_{j_1}}\dot{x}_{\ell_{j_2}}^{i_{j_2}}\cdots\dot{x}_{\ell_{j_r}}^{i_{j_r}}$. For all of the remaining Lyndon words ℓ_{a_1}, ℓ_{a_2}, $\ldots$, $\ell_{a_{k-r}}$ with length greater than 1 we define the Lie portion of the word to be $L(w) = P_{\ell_{a_1}^{i_{a_1}}\ell_{a_2}^{i_{a_2}}\cdots\ell_{a_{k-r}}^{i_{a_{k-r}}}}$. We will define the hybrid elements to be $H_w := M(w)L(w) = \Phi(X(w)\otimes L(w))$.

The result of this section is:

Theorem 10. *The noncommutative polynomials H_w are triangularly related to and expand positively in the P_u basis. Precisely, for w of length r,*

$$H_w = r!\,P_w + \textit{terms } c_u P_u \textit{ with } u \textit{ lexicographically smaller than } w.$$

As a consequence, the set $\{H_w\}_{w\in[n]^r}$ is a basis for the noncommutative polynomials of $\mathbb{Q}\langle X_n\rangle$ of degree r.

We require a few facts about Lyndon words and the lexicographic ordering which can be found in [12].

(1) If u and v are Lyndon words and $u < v$ then uv is a Lyndon word. ([12], (5.1.2))

(2) If $u < v$ and u is not a prefix of v, then $ux < vy$ for all words x, y. ([12], Lemma 5.2.(i))
(3) If $w = \ell_1\ell_2\cdots\ell_k$ with $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ then ℓ_k is the smallest (with respect to the $>$ order) nontrivial suffix of w. ([12], Lemma 7.14)
(4) If $\ell' < \ell$, are both Lyndon words, then $\ell'\ell < \ell\ell'$ (follows from (1)). As a consequence, for $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$, $\ell_1\ell_2\cdots\ell_k \geq \ell_{\sigma(1)}\ell_{\sigma(2)}\cdots\ell_{\sigma(k)}$ for any permutation $\sigma \in \mathfrak{S}_k$ with equality if and only $\ell_i = \ell_{\sigma(i)}$ for all $1 \leq i \leq k$.

Proof. To see that (4) holds consider a weakly decreasing product of Lyndon words $\ell_1\ell_2\cdots\ell_k$. If $id \to \sigma^{(1)} \to \sigma^{(2)} \to \cdots \to \sigma$ is a chain in the weak right order then we have just shown that

$$\ell_{\sigma^{(i)}(1)}\ell_{\sigma^{(i)}(1)}\cdots\ell_{\sigma^{(i)}(k)} \geq \ell_{\sigma^{(i+1)}(1)}\ell_{\sigma^{(i+1)}(2)}\cdots\ell_{\sigma^{(i+1)}(k)}$$

with equality if and only if the two Lyndon factors which were transposed are equal. Therefore there exists a chain of words one greater than or equal to the next with $\ell_1\ell_2\cdots\ell_k$ on one end and $\ell_{\sigma(1)}\ell_{\sigma(2)}\cdots\ell_{\sigma(k)}$ on the other. □

Theorem 10 will be established after the following lemma.

Lemma 11. *Let w be a word and $\ell_1\ell_2\cdots\ell_r$ the decomposition of w into a decreasing product of Lyndon words. Let ℓ be a Lyndon word such that $\ell = af_1f_2\cdots f_k$ with a one of the variables, each f_i a Lyndon word and $f_i \geq f_{i+1}$ and $f_k \geq \ell_1$. Let $u = \ell_1\cdots\ell_d\ell\ell_{d+1}\cdots\ell_r$ where $\ell_d > \ell \geq \ell_{d+1}$ or $d = 0$ and $\ell \geq \ell_1$. Then*

$$P_\ell P_w = P_u$$
$$\quad + \textit{terms } c_vP_v \textit{ where } v \textit{ is lexicographically smaller than } u \textit{ and } c_v \geq 0.$$

Proof. Assume that $r = 1$, and we have that either $\ell \geq \ell_1$ and $P_\ell P_{\ell_1} = P_{\ell\ell_1}$ and we are done, or $\ell < \ell_1$ and

$$P_\ell P_{\ell_1} = P_{\ell_1}P_\ell + [P_\ell, P_{\ell_1}].$$

In this case $P_{\ell_1}P_\ell = P_{\ell_1\ell}$. By (1) we know that $\ell\ell_1$ is Lyndon. Moreover, $\ell\ell_1$ is its standard factorization (this follows from (3), since the nontrivial suffixes of $\ell\ell_1$ are all suffixes of $f_1f_2\cdots f_k\ell_1$, which is a nonincreasing product of Lyndon words). Therefore $P_{\ell\ell_1} = [P_\ell, P_{\ell_1}]$ and $P_\ell P_{\ell_1} = P_{\ell_1\ell} + P_{\ell\ell_1}$. By (4), $\ell\ell_1 < \ell_1\ell$ so the triangularity relation holds.

Now for an arbitrary $r > 1$ we have the same two cases. Either $\ell \geq \ell_1$ and $P_\ell P_{\ell_1}P_{\ell_2}\cdots P_{\ell_r} = P_{\ell w}$, or $\ell < \ell_1$ and

$$P_\ell P_{\ell_1}P_{\ell_2}\cdots P_{\ell_r} = P_{\ell_1}P_\ell P_{\ell_2}\cdots P_{\ell_r} + [P_\ell, P_{\ell_1}]P_{\ell_2}\cdots P_{\ell_r}.$$

Our induction hypothesis holds for $P_\ell P_{\ell_2}\cdots P_{\ell_r}$ since $f_k \geq \ell_1 \geq \ell_2$, hence $P_\ell P_{\ell_2}\cdots P_{\ell_r} = P_{u'} + \sum_{v'<u'} c'_{v'}P_{v'}$ where $u' = \ell_2\cdots\ell_d\ell\ell_{d+1}\cdots\ell_r$. Moreover, $P_{\ell_1}P_{u'} = P_{\ell_1u'} = P_u$ since $\ell_1 \geq \ell_2$ and $P_{\ell_1}P_{v'} = P_{\ell_1v'}$ since $\ell_1 \geq$ any Lyndon prefix of v'.

Since $[P_\ell, P_{\ell_1}] = P_{\ell\ell_1}$ by (3), and $\ell_1 \geq \ell_2$, we have by the induction hypothesis that $P_{\ell\ell_1} P_{\ell_2} \cdots P_{\ell_r} = P_{u''} + \sum_{v''<u''} c''_{v''} P_{v''}$ where

$$u'' = \ell_2 \cdots \ell_{d'} \ell\ell_1 \ell_{d'+1} \cdots \ell_r$$

with $\ell_{d'} > \ell\ell_1 \geq \ell_{d'+1}$. In order to justify the induction step we also need to have that $u'' < u$. This follows from (4) since u'' is a permutation of the factors of u and $\ell_1 > \ell$ and ℓ lies to the left of ℓ_1 in u'. $\square$

We are now in a position to prove Theorem 10.

Proof. H_w is defined as the product $M(w)L(w)$ where $M(w)$ is a a shuffle of monomials. It expands as $M(w) = \sum \tilde{c}_b \mathbf{x}_b$ with $\sum \tilde{c}_b = r!$ and where each monomial in $M(w)$ has the same number of x_1s, x_2s, etc. We fix one such monomial $\mathbf{x}_b$ that as follows by indexing its letters backwards: $\mathbf{x}_b = x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$. We define inductively words $w[k], \ldots, w[1], w[0]$ as follows: $w[k] := w$ and $w[j-1]$ is the word obtained from $w[j]$ by removing one of its Lyndon factors of length 1 equal to x_{i_j}. Remark that $L(w[j]) = L(w)$ for all j. Then we establish by induction on j that

$$\begin{aligned} x_{i_j} x_{i_{j-1}} \cdots x_{i_1} L(w) = P_{w[j]} \\ + \text{ terms } c_v P_v \text{ with } v \text{ lexicographically smaller than } w[j] \end{aligned}$$

by applying Lemma 11 with x_{i_j} for ℓ and $w[j]$ for w. $\square$

References

[1] N. Bergeron, C. Reutenauer, M. Rosas, M. Zabrocki, Invariants and Coinvariants of the Symmetric Group in Noncommuting Variables, to appear in Canad. J. Math.

[2] N. Bergeron and M. Zabrocki, The Hopf algebra of symmetric functions in non-commutative variables is free and cofree. Preprint, arXiv: math.CO/0509265.

[3] C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math. 77 (1955), 778–782.

[4] V. Drensky, Lattices of varieties of associative algebras. Serdica 8 (1982), 20–31.

[5] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. Algebra 91:1 (1984), 1–17.

[6] W. Fulton, Young tableaux. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[7] H. P. Kraft and C. Procesi, Classical Invariant Theory: A primer, lecture notes, `http://www.math.unibas.ch/~kraft/Papers/KP-Primer.pdf`.

[8] C. Lenormand, Opérateurs sur les polynômes non–commutatifs: définitions et notations, Séminaire Schützenberger–Lentin–Nivat (Problèmes mathématiques de la théorie des automates), 1969–1970, exposé num. 3, 9 p.

[9] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, The Clarendon Press, Oxford University Press, 1995.

[10] A. P. Popov, Identities of the Tensor Square of the Grassmann Algebra, Algebra and Logic, 21 (1982) 4, pp. 296-316.

[11] A. P. Popov, Module Structure of Space of Proper Polynomials of Degree Seven, C. R. Acad. Bulgare, 38:3 (1985), 295–298.

[12] C. Reutenauer, Free Lie algebras. London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[13] M. Rosas, B. Sagan, Symmetric Functions in Noncommuting Variables. Transactions of the American Mathematical Society, 358:1 (2006), 215–232.
[14] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/.
[15] H. Wilf, Generatingfunctionology. Second edition. Academic Press, Inc., Boston, MA, 1994.
[16] M. C. Wolf, Symmetric functions of noncommutative elements, Duke Math. J. 2 (1936), 626–637.

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